Properties of Empirical Mode Decomposition

Marta Campi [1], Gareth W. Peters [1], Nourddine Azzaoui [2] and Tomoko Matsui [3]

[1] University College London (UCL) - London, UK

[2]Laboratoire de Mathemtiques Blaise Pascal (LBMP) - Clermont Ferrand, France

[3]The Institute of Statistical Mathematics (ISM) - Tokyo, Japan

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Research Organization of Information and Systems
The Institute of Statistical Mathematics

Outline

1. Classical Stationary Methods
2. Empirical Mode Decomposition
3. Constructing EMD
4. A speech analysis application
5. Conclusion

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Stochastic process and path realization

A univariate real-valued stochastic process defined as $\{\dots X_{t_k-h}, \dots, X_{t_1}, X_{t_2}, \dots X_{t_k+h}, \dots\} = \{X_t\}_{t=-\infty}^{\infty}$ for all $k \in \mathbb{N}, h \in \mathbb{R}$ is a sequence of random variables indexed by time t, with finite path realization given by $\{X_0 = x_0, X_1 = x_1, X_2 = x_2, \dots, X_t = x_t\} = \{x_t\}_{t=0}^{T}$

The classical approach to investigate such processes is given by considering the following structure:

$$X_t \stackrel{d}{=} m_t + s_t + Y_t$$

Strict Stationarity

A stochastic process $\{X_t\}$ is said to be strictly stationary if:

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_k}) \stackrel{d}{=} (X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_k+h})$$

Covariance Stationarity

A stochastic process $\{X_t\}$ is said to be weakly stationary (or covariance stationary) if:

$$\mathbb{E}[X_t] = m \quad \text{for all} \quad t \in T$$

$$\mathbb{E}[X_t^2] < \infty \quad \text{for all} \quad t \in T$$

$$Cov(X_t, X_{t+h}) = \gamma(h), \quad h \in T \quad \text{such that} \quad t + h \in T$$

Classical Stationary Methods

Which methods are available to decompose and analyse such stochastic processes?

Deterministic time

- Splines
- Polynomial interpolation

Stochastic time

- ARMA model
- ARIMA model
- Etc.

Deterministic time-frequency

- Fast Fourier Transform
- Discrete Fourier Transform

Stochastic time-frequency

- Fourier transform
- Wavelet Transform

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Fourier transform

$$h(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi\omega t} dt$$

where $h(\omega) = \mathcal{F}[x(t)]$ is such that $h(\omega) : \mathbb{R} \to \mathbb{C}$ for any $\omega \in \mathbb{R}$.

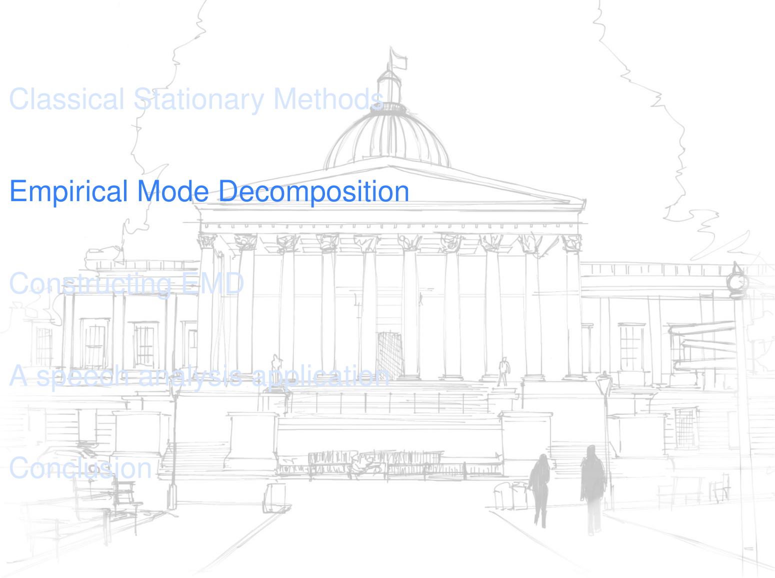
Wavelet transform

$$CWT(a, \tau; x, \psi) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} x(t) \psi^* \left(\frac{t - \tau}{a} \right) dt$$

- **infinite number** of basis
- **parametric** structure
- **a-priori** basis
- **stationarity** or **linearity** of the system

What happens if some of these assumptions do not hold?

Outline



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Empirical Mode Decomposition

Given an observed time series $\{x_t\}_{t=0}^{T}$, we construct a continuous interpolation representation $\tilde{x}(t)$, such that $\tilde{x}(t) \in C^1[0, T]$.

Oscillation of $\tilde{x}(t)$

Given $\{\tau_i\}_{i=0}^{T}$ such that $0 = \tau_0 < \tau_1 < \cdots < \tau_T = T$, we define the oscillation over each interval $\tilde{x}(\tau_i, \tau_{i+1}]$ according to:

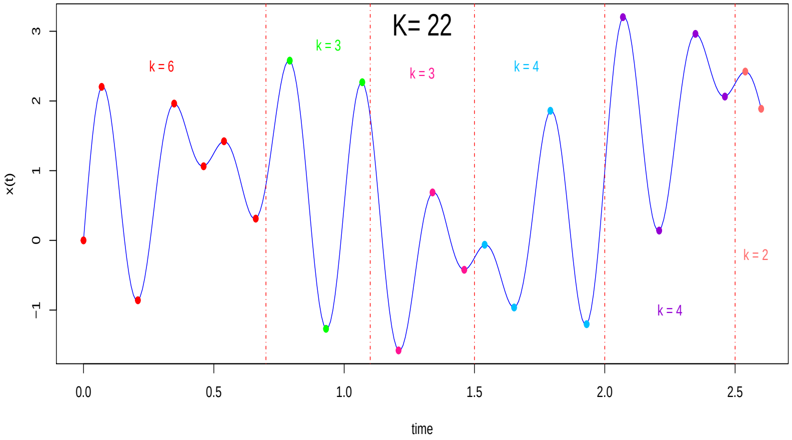
$$osc(\tilde{x}(\tau_i, \tau_{i+1}]) = \left| \left\{ \frac{d}{dt}\tilde{x}(t) = 0 : t \in (\tau_i, \tau_{i+1}], \frac{d}{dt}\tilde{x}(t) \neq 0 \forall t \right\} \right| = k_i < \infty$$

with $i = 0, \ldots, T$.

Remark: NO assumptions on stationarity for $\tilde{x}(t)$.

Let $K = \sum_{i=0}^{T} k_i$ the number of turning points over $[0, T]$.

Empirical Mode Decomposition



Can we find a finite basis decomposition admitting meaningful time and frequency domain interpretation in such a framework?
Ans: YES { N. Huang et al., 1998 }

Empirical Mode Decomposition

$\tilde{x}(t)$ can be decomposed as:

$$\tilde{x}(t) = \sum_{k=1}^{K} c_k(t) + r(t) = \left(\sum_{k=1}^{K+1} c_k(t), \quad c_{k+1}(t) = r(t) \right)$$

where each $c_k(t)$ is called **Intrinsic Mode Function** and $r(t)$ is a final tendency or **residual**.

Define the interpolations of maxima and minima of $\tilde{x}(t)$ as **upper envelope** $M(t)$ and **lower envelope** $m(t)$ respectively and the **mean envelope** as $d(t) = \frac{M(t)+m(t)}{2}$.

Empirical Mode Decomposition

The IMF basis $\{c_k(t)\}_{k=1}^{K}$ are designed to satisfy the following two conditions:

- *Local symmetry* At any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is zero.
- *Oscillations* The number of extrema and the number of zero-crossings must either equal or differ at most by one.

$$M(t) > \tilde{x}(t) \quad \text{and} \quad \tilde{x}(t) < m(t) \quad \forall t \quad \text{except} \quad \tau_0, \tau_1, \ldots, \tau_T$$

The residual $r(t)$ is designed to satisfy the following condition:

The residual is a curve with at most one extremum.

$$osc(r(t)) \in \{0, 1\}$$
$$r'(t) \geq 0 \quad \text{or} \quad r'(t) < 0 \quad \text{over} \quad [0, T]$$

Instantaneous frequency

$$f_k(t) = \frac{1}{2\pi}\frac{d\theta_k(t)}{dt}$$

An analytical signal is defined as $z_k(t) = c_k(t) + jy_k(t)$ or $z_k(t) = a_k(t)e^{j\theta_k(t)}$ where

- $a_k(t)$ is the **amplitude** of $z(t)$
- $\theta_k(t) = arctan\frac{y_k(t)}{c_k(t)}$ is **the instantaneous phase.**

Recall: $\tilde{x}(t) = \sum_{k=1}^{K} c_k(t) + r(t)$

Hilbert transform (Cauchy Principal Value Integral)

$$y_k(t) = -\frac{1}{\pi}\lim_{\epsilon\to\infty}\int_{-\epsilon}^{+\epsilon}\frac{c_k(t+\tau) - c_k(t-\tau)}{t}d\tau$$

This concept makes sense when the signal $z_k(t)$ is almost circular within the complex domain.

The EMD provides such property, therefore after the EMD and the Hilbert transform each IMF $c_k(t)$ and $\tilde{x}(t)$ are expressed respectively as:

$$c_k(t) = Re\left\{a_k(t)exp\left(j\int 2\pi f_k(t)dt\right)\right\}$$

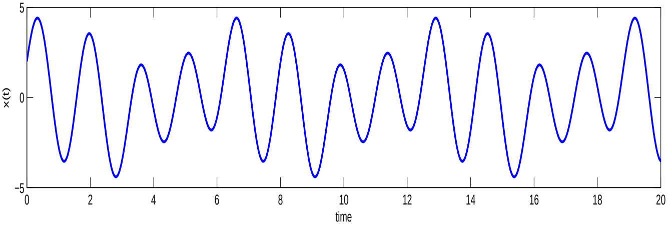
$$\tilde{x}(t) = Re\left\{\sum_{k=1}^{K+1} a_k(t)exp\left(j\int 2\pi f_k(t)dt\right)\right\}$$

Outline

Constructing EMD

1. **Step 1** Find local extrema of $\tilde{x}(t)$
2. **Step 2** Compute the upper envelope $M(t)$ and the lower envelope $m(t)$ by employing spline interpolations (cubic, akima, b-spline, etc.)
3. **Step 3** Update the signal $\tilde{x}(t) \leftarrow \tilde{x}(t) - \frac{M(t)+m(t)}{2}$
4. **Step 4** Repeat 1, 2 and 3 until achieving an IMF $c_k(t)$
5. **Step 5** Subtract the obtained IMF from the signal $\tilde{x}(t) \leftarrow \tilde{x}(t) - c_k(t)$
6. **Step 6** Repeat 1-5 until achieving a tendency

Constructing EMD



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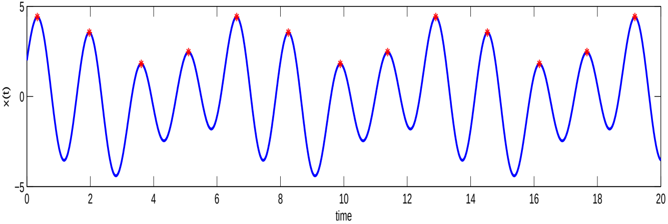

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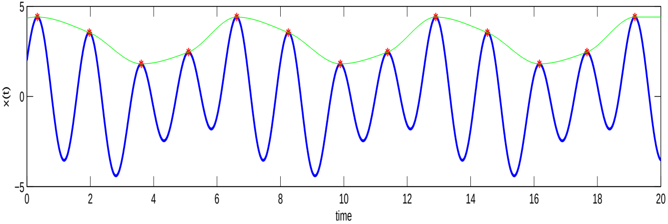

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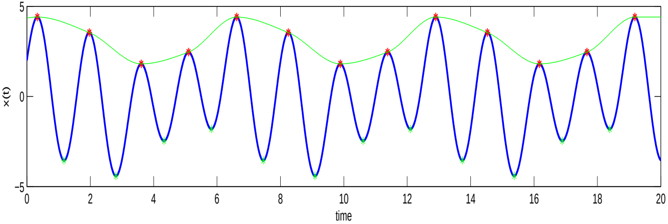

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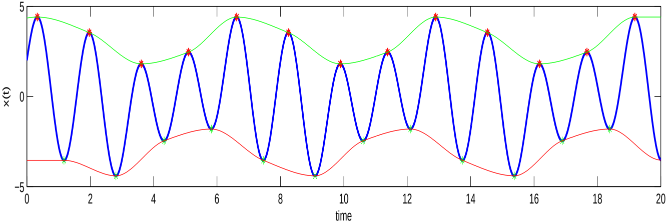

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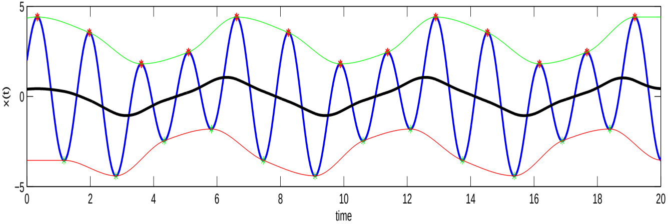

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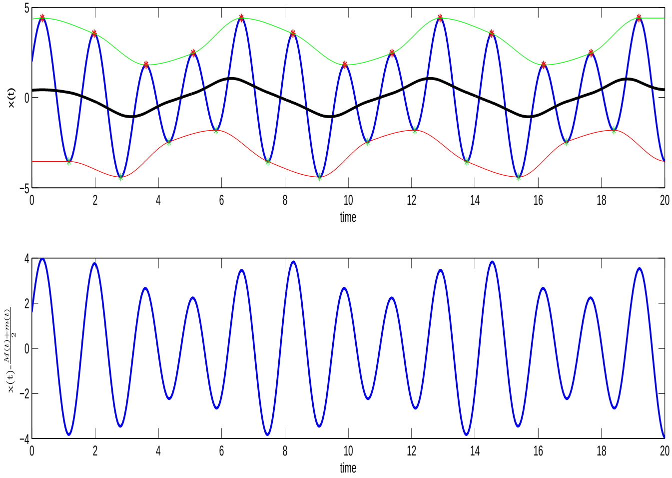

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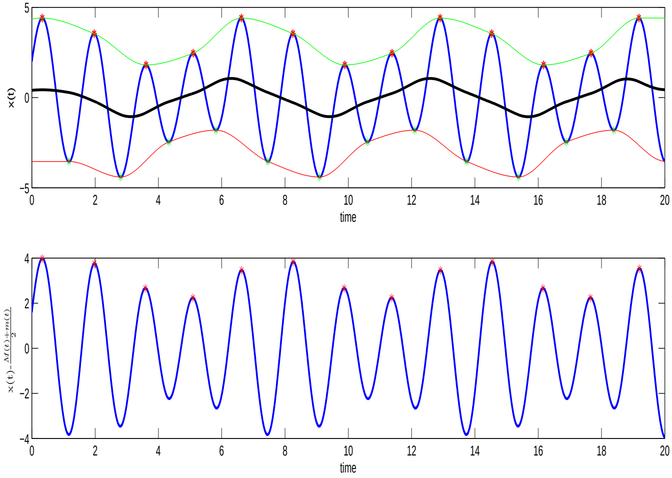

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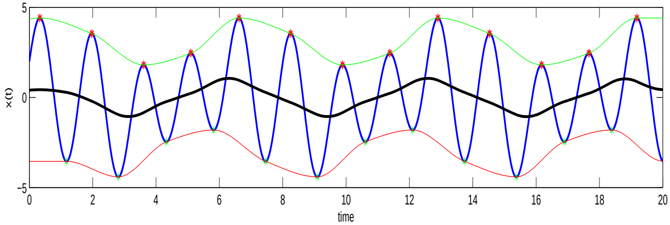

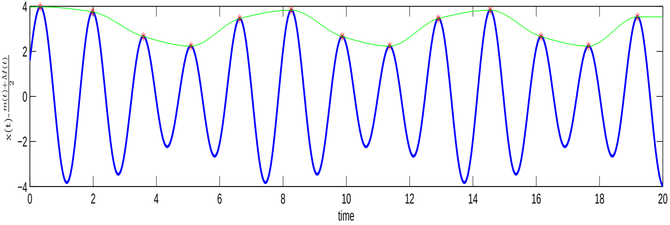

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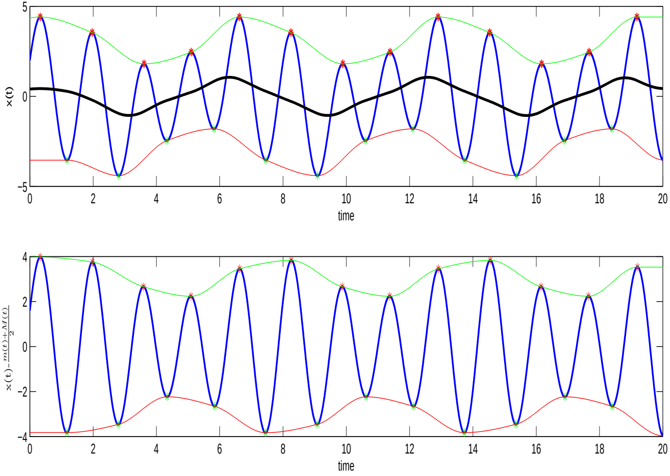

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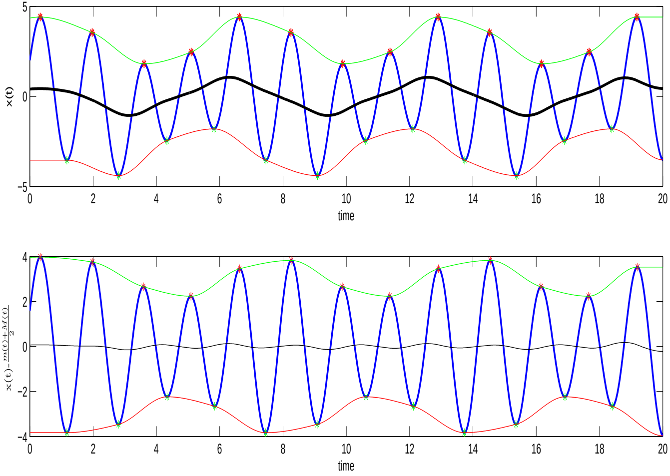

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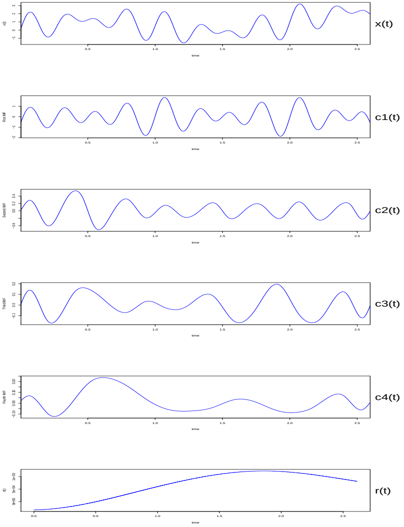

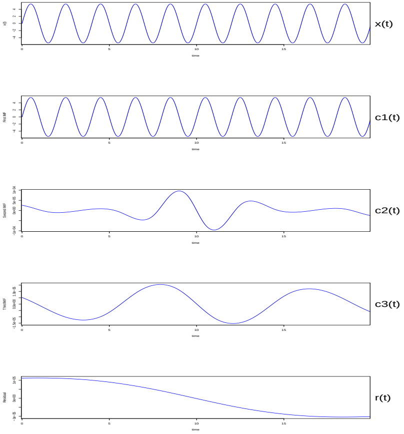

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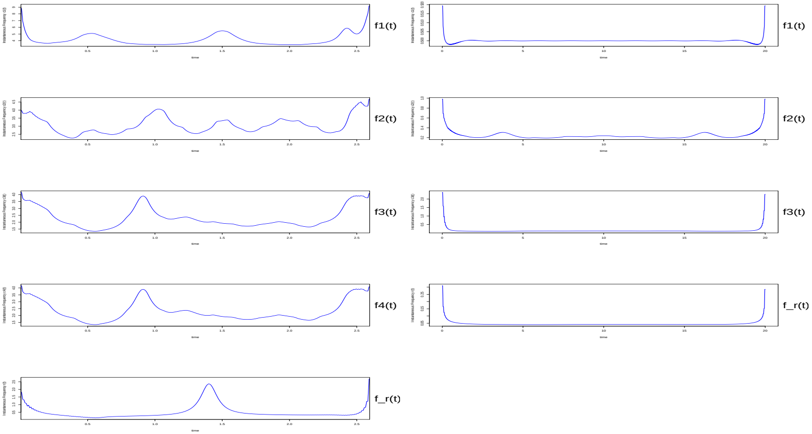
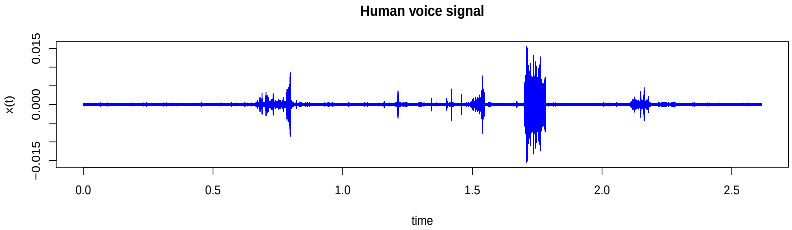
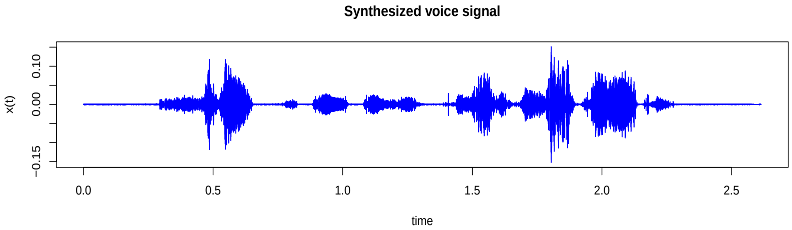


Outline

A speech analysis application





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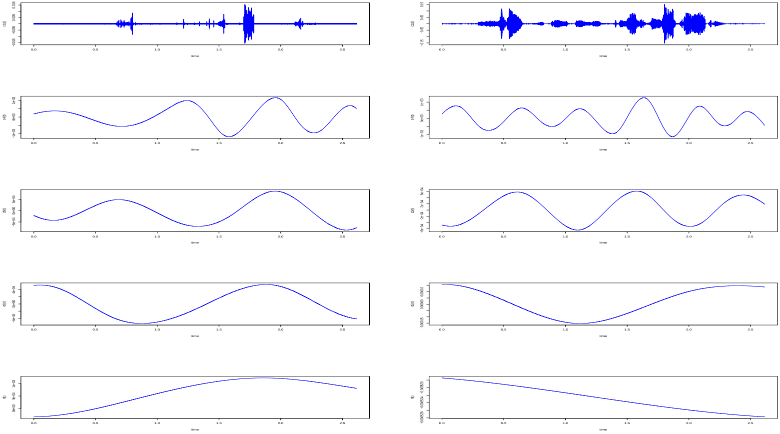

A speech analysis application

Classifier features:

- **Intrinsic Mode Functions**: $c_1(t)$, $c_2(t)$, ... , $r(t)$.
- **Instantaneous frequencies**: $f_1(t)$, $f_2(t)$, ... , $f_r(t)$.
- **Coefficients splines** used to represent the IMFs.
- **Classical Statistics.**

Outline

Conclusion

- Exploiting an a posteriori time-frequency decomposition for **non-stationary and non-linear** systems with minimal restrictions.
- A close form of the Huang-Hilbert transform leading to **a close form of the instantaneous frequency**.
- Study of the **constructing algorithm** and its performances according to its heuristic rules (future research aimed to improve it).
- Using such a decomposition for features extraction and classification.
- A speech analysis application.